



IIT
GURUKUL
PRACTICE CENTER

BEST
IIT & JEE MATERIAL
Revised as per CBSE

MATHS

CLASS 11

THEORY



Contact

 9004567771, 8999978871

 igurukul@gmail.com

 <https://igurukul.in/>

 1001, Kamdhenu, Commerze Kharghar,
Nerul Seawood



@IIT_GURUKUL_OFFICIAL

IIT GURUKUL PRACTICE CENTER

BINOMIAL THEOREM

BINOMIAL EXPRESSION

Any algebraic expression which contains two dissimilar terms is called Binomial expression.

For example : $x - y, xy + \frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{1/3}} + 3$ etc.

Terminology Used in Binomial Theorem

Factorial notation : $n!$ or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)\dots\dots 3.2.1 & ; \text{ if } n \in \mathbb{N} \\ 1 & ; \text{ if } n = 0 \end{cases}$$

❖ $n! = n \cdot (n-1)! ; n \in \mathbb{N}$

BINOMIAL THEOREM

The formula by which any positive integral power of a Binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM**.

If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then : $(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n y^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r$

This theorem can be proved by induction.

(a) The number of terms in the expansion is $(n+1)$ i.e. one more than the index.
 (b) The sum of the indices of x & y in each term is n .
 (c) The Binomial coefficients of the terms (${}^nC_0, {}^nC_1, \dots$) equidistant from the beginning and the end are equal.
 i.e. ${}^nC_r = {}^nC_{n-r}$

(d) Symbol nC_r can also be denoted by $\binom{n}{r}, C(n, r)$ or A_r^n .

❖ The coefficient of x^r in $(1+x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r \cdot {}^nC_r$

Some Important Expansions :

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$.
 (ii) $(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \dots + (-1)^n \cdot {}^nC_n x^n$.

Ex. Expand the following Binomials :

(i) $(x - 3)^5$ (ii) $\left(1 - \frac{3x^2}{2}\right)^4$

Sol. (i) $(x - 3)^5 = {}^5C_0 x^5 + {}^5C_1 x^4 (-3)^1 + {}^5C_2 x^3 (-3)^2 + {}^5C_3 x^2 (-3)^3 + {}^5C_4 x (-3)^4 + {}^5C_5 (-3)^5$
 $= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

(ii) $\left(1 - \frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1 \left(-\frac{3x^2}{2}\right) + {}^4C_2 \left(-\frac{3x^2}{2}\right)^2 + {}^4C_3 \left(-\frac{3x^2}{2}\right)^3 + {}^4C_4 \left(-\frac{3x^2}{2}\right)^4$
 $= 1 - 6x^2 + \frac{27}{2} x^4 - \frac{27}{2} x^6 + \frac{81}{16} x^8$

MATHS FOR JEE MAINS & ADVANCED

Ex. Find the value of $\frac{(18^3 + 7^3 + 3 \cdot 18 \cdot 7 \cdot 25)}{3^6 + 6 \cdot 243 \cdot 2 + 15 \cdot 81 \cdot 4 + 20 \cdot 27 \cdot 8 + 15 \cdot 9 \cdot 16 + 6 \cdot 3 \cdot 32 + 64}$

Sol. The numerator is of the form $a^3 + b^3 + 3ab(a+b) = (a+b)^3$

Where, $a = 18$ and $b = 7$ $\therefore N^r = (18+7)^3 = (25)^3$

Denominator can be written as

$$3^6 + {}^6C_1 \cdot 3^5 \cdot 2^1 + {}^6C_2 \cdot 3^4 \cdot 2^2 + {}^6C_3 \cdot 3^3 \cdot 2^3 + {}^6C_4 \cdot 3^2 \cdot 2^4 + {}^6C_5 \cdot 3 \cdot 2^5 + {}^6C_6 \cdot 2^6 = (3+2)^6 = 5^6 = (25)^3$$

$$\therefore \frac{N^r}{D^r} = \frac{(25)^3}{(25)^3} = 1$$

Pascal's Triangle

A triangular arrangement of numbers as shown. The numbers give the coefficients for the expansion of $(x+y)^n$. The first row is for $n=0$, the second for $n=1$, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & 1 & 1 & & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & 3 & 3 & 1 & & & \\ & 1 & 4 & 6 & 4 & 1 & & & \\ 1 & 5 & 10 & 10 & 5 & 1 & & & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & & \\ & & & & & & & & \text{etc.} \end{array}$$

IMPORTANT TERMS IN THE BINOMIAL EXPANSION

(a) General Term :

The general term or the $(r+1)^{\text{th}}$ term in the expansion of $(x+y)^n$ is given by

$$T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r$$

Ex. Find (i) 28th term of $(5x+8y)^{30}$ (ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

Sol. (i) $T_{27+1} = {}^{30}C_{27} (5x)^{30-27} (8y)^{27} = \frac{30!}{3!27!} (5x)^3 \cdot (8y)^{27}$

(ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$$T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(-\frac{5}{2x}\right)^6 = \frac{9!}{3!6!} \left(\frac{4x}{5}\right)^3 \left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$$

(b) Middle Term :

The middle term(s) in the expansion of $(x+y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{\frac{(n+2)}{2}} = {}^nC_{\frac{n}{2}} \cdot x^{n/2} \cdot y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{\frac{(n+1)}{2}}$ & $T_{\frac{(n+1)}{2}+1}$

Middle term has greatest Binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$$\Rightarrow \quad {}^n C_r \text{ will be maximum } \begin{cases} \text{When } r = \frac{n}{2} \text{ if } n \text{ is even} \\ \text{When } r = \frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ if } n \text{ is odd} \end{cases}$$

\Rightarrow The term containing greatest Binomial coefficient will be middle term in the expansion of $(1 + x)^n$

Ex. Find the middle term(s) in the expansion of

(i) $\left(1 - \frac{x^2}{2}\right)^{14}$ (ii) $\left(3a - \frac{a^3}{6}\right)^9$

Sol. (i) $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{\text{th}}$ term.

It means T_8 is middle term

$$T_8 = {}^{14}C_7 \left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16} x^{14}.$$

(ii) $\left(3a - \frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{\text{th}}$ & $\left(\frac{9+1}{2} + 1\right)^{\text{th}}$.

It means T_5 & T_6 are middle terms

$$T_5 = {}^9C_4 (3a)^{9-4} \left(-\frac{a^3}{6}\right)^4 = \frac{189}{8} a^{17}$$

$$T_6 = {}^9C_5 (3a)^{9-5} \left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16} a^{19}.$$

(c) Term Independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Ex. Find the term independent of x in $\left[\sqrt{\frac{x}{3}} + \sqrt{\frac{3}{2x^2}}\right]^{10}$.

Sol. General term in the expansion is

$${}^{10}C_r \left(\frac{x}{3}\right)^{\frac{r}{2}} \left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10} \cdot \frac{3^{5-r}}{2^{\frac{10-r}{2}}} \text{ For constant term, } \frac{3r}{2} = 10 \Rightarrow r = \frac{20}{3}$$

which is not an integer. Therefore, there will be no constant term.

(d) Numerically Greatest Term :

Binomial expansion of $(a + b)^n$ is as follows : –

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_r a^{n-r} b^r + \dots + {}^nC_n a^0 b^n$$

If we put certain values of a and b in RHS, then each term of Binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{\text{th}}$ terms respectively

$$T_r = {}^nC_{r-1} a^{n-(r-1)} b^{r-1}$$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$\text{Now, } \left| \frac{T_{r+1}}{T_r} \right| = \left| \frac{{}^nC_r a^{n-r} b^r}{{}^nC_{r-1} a^{n-(r-1)} b^{r-1}} \right| = \frac{n-r+1}{r} \cdot \left| \frac{b}{a} \right|$$

$$\text{Consider } \left| \frac{T_{r+1}}{T_r} \right| \geq 1$$

$$\left(\frac{n-r+1}{r} \right) \left| \frac{b}{a} \right| \geq 1$$

$$\frac{n+1}{r} - 1 \geq \left| \frac{a}{b} \right|$$

$$r \leq \frac{n+1}{1 + \left| \frac{a}{b} \right|}$$

Case - I When $\frac{n+1}{1 + \left| \frac{a}{b} \right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ ($r = 1, 2, 3, \dots, m-1$)

i.e. $T_2 > T_1, T_3 > T_2, \dots, T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ ($r = m+1, m+2, \dots, n$)

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \dots, T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1 + \left| \frac{a}{b} \right|}$ is an integer, say m , then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When $\frac{n+1}{1 + \left| \frac{a}{b} \right|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \frac{n+1}{1 + \left| \frac{a}{b} \right|}$ ($r = 1, 2, 3, \dots, m-1, m$)

i.e. $T_2 > T_1, T_3 > T_2, \dots, T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \frac{n+1}{1 + \left| \frac{a}{b} \right|}$ ($r = m + 1, m + 2, \dots, n$)

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \dots, T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1 + \left| \frac{a}{b} \right|}$ is not an integer and its integral part is m , then T_{m+1} will be the numerically greatest term.

(i) In any Binomial expansion, the middle term(s) has greatest Binomial coefficient.

In the expansion of $(a + b)^n$

n	No. of Greatest Binomial Coefficient	Greatest Binomial Coefficient
Even	1	${}^n C_{n/2}$
Odd	2	${}^n C_{(n-1)/2}$ and ${}^n C_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put $a = b = 1$, and proceed as discussed above.

Ex. Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \frac{1}{5}$

Sol. Using $\frac{n+1}{1 + \left| \frac{a}{b} \right|} - 1 \leq r \leq \frac{n+1}{1 + \left| \frac{a}{b} \right|}$

$$\frac{11+1}{1 + \left| \frac{3}{-5x} \right|} - 1 \leq r \leq \frac{11+1}{1 + \left| \frac{3}{-5x} \right|}$$

Solving we get $2 \leq r \leq 3$

$\therefore r = 2, 3$

So, the greatest terms are T_{2+1} and T_{3+1} .

\therefore Greatest term (when $r = 2$)

$T_3 = {}^{11}C_2 \cdot 3^9 \cdot (-5x)^2 = 55 \cdot 3^9 = T_4$

From above we say that the value of both greatest terms are equal.

Ex. If n is positive integer, then prove that the integral part of $(7 + 4\sqrt{3})^n$ is an odd number.

Sol. Let $(7 + 4\sqrt{3})^n = I + f$ (i)

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now, $0 < 7 - 4\sqrt{3} < 1$

$0 < (7 - 4\sqrt{3})^n < 1$

Let $(7 - 4\sqrt{3})^n = f'$ (ii)

$\Rightarrow 0 < f' < 1$

Adding (i) and (ii)

$$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$= 2 [{}^n C_0 7^n + {}^n C_2 7^{n-2} (4\sqrt{3})^2 + \dots]$$

$I + f + f' = \text{even integer} \Rightarrow (f + f' \text{ must be an integer})$

$0 < f + f' < 2 \Rightarrow f + f' = 1$

$I + 1 = \text{even integer}$

therefore I is an odd integer.

Ex. What is the remainder when 5^{99} is divided by 13.

Sol. $5^{99} = 5 \cdot 5^{98} = 5 \cdot (25)^{49} = 5 (26 - 1)^{49}$

$$= 5 [{}^{49} C_0 (26)^{49} - {}^{49} C_1 (26)^{48} + \dots + {}^{49} C_{48} (26)^1 - {}^{49} C_{49} (26)^0]$$

$$= 5 [{}^{49} C_0 (26)^{49} - {}^{49} C_1 (26)^{48} + \dots + {}^{49} C_{48} (26)^1 - 1]$$

$$= 5 [{}^{49} C_0 (26)^{49} - {}^{49} C_1 (26)^{48} + \dots + {}^{49} C_{48} (26)^1 - 13] + 60$$

$$= 13(k) + 52 + 8 \text{ (where k is a positive integer)}$$

$$= 13(k+4) + 8$$

Hence, remainder is 8.

Some Standard Expansions

(i) Consider the expansion

$$(x + y)^n = \sum_{r=0}^n {}^n C_r x^{n-r} y^r = {}^n C_0 x^n y^0 + {}^n C_1 x^{n-1} y^1 + \dots + {}^n C_r x^{n-r} y^r + \dots + {}^n C_n x^0 y^n \quad \dots \text{(i)}$$

(ii) Now replace $y \rightarrow -y$ we get

$$(x - y)^n = \sum_{r=0}^n {}^n C_r (-1)^r x^{n-r} y^r = {}^n C_0 x^n y^0 - {}^n C_1 x^{n-1} y^1 + \dots + {}^n C_r (-1)^r x^{n-r} y^r + \dots + {}^n C_n (-1)^n x^0 y^n \quad \dots \text{(ii)}$$

(iii) Adding (i) & (ii), we get

$$(x + y)^n + (x - y)^n = 2[{}^n C_0 x^n y^0 + {}^n C_2 x^{n-2} y^2 + \dots]$$

(iv) Subtracting (ii) from (i), we get

$$(x + y)^n - (x - y)^n = 2[{}^n C_1 x^{n-1} y^1 + {}^n C_3 x^{n-3} y^3 + \dots]$$

PROPERTIES OF BINOMIAL COEFFICIENTS

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n = \sum_{r=0}^n {}^n C_r x^r ; n \in \mathbb{N} \quad \dots \text{(i)}$$

where $C_0, C_1, C_2, \dots, C_n$ are called combinatorial (Binomial) coefficients.

(a) The sum of all the Binomial coefficients is 2^n .

Put $x = 1$, in (i) we get

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n \Rightarrow \sum_{r=0}^n {}^n C_r = 2^n \quad \dots \text{(ii)}$$

(b) Put $x = -1$ in (i) we get

$$C_0 - C_1 + C_2 - C_3 + \dots + C_n = 0 \Rightarrow \sum_{r=0}^n (-1)^r {}^n C_r = 0 \quad \dots \text{(iii)}$$

(c) The sum of the Binomial coefficients at odd position is equal to the sum of the Binomial coefficients at even position and each is equal to 2^{n-1} .

From (ii) & (iii), $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

(d) ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$

(e) $\frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r}$

(f) ${}^n C_r = \frac{n}{r} \cdot {}^{n-1} C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2} C_{r-2} = \dots = \frac{n(n-1)(n-2)\dots(n-r+1)}{r(r-1)(r-2)\dots 1}$

(g) ${}^n C_r = \frac{r+1}{n+1} \cdot {}^{n+1} C_{r+1}$

Ex. Prove that : ${}^{25} C_{10} + {}^{24} C_{10} + \dots + {}^{10} C_{10} = {}^{26} C_{11}$

Sol. LHS = ${}^{10} C_{10} + {}^{11} C_{10} + {}^{12} C_{10} + \dots + {}^{25} C_{10}$
 $\Rightarrow {}^{11} C_{11} + {}^{11} C_{10} + {}^{12} C_{10} + \dots + {}^{25} C_{10}$
 $\Rightarrow {}^{12} C_{11} + {}^{12} C_{10} + \dots + {}^{25} C_{10}$
 $\Rightarrow {}^{13} C_{11} + {}^{13} C_{10} + \dots + {}^{25} C_{10}$

and so on.

\therefore LHS = ${}^{26} C_{11}$

Aliter

LHS = coefficient of x^{10} in $\{(1+x)^{10} + (1+x)^{11} + \dots + (1+x)^{25}\}$

\Rightarrow coefficient of x^{10} in $\left[(1+x)^{10} \frac{(1+x)^{16} - 1}{1+x-1} \right]$

\Rightarrow coefficient of x^{10} in $\frac{[(1+x)^{26} - (1+x)^{10}]}{x}$

\Rightarrow coefficient of x^{11} in $[(1+x)^{26} - (1+x)^{10}] = {}^{26} C_{11} - 0 = {}^{26} C_{11}$

Ex. Prove that :

(i) $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$

(ii) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

Sol. (i) L.H.S. = $\sum_{r=1}^n r \cdot {}^n C_r = \sum_{r=1}^n r \cdot \frac{n}{r} \cdot {}^{n-1} C_{r-1}$

$= n \sum_{r=1}^n {}^{n-1} C_{r-1} = n \cdot [{}^{n-1} C_0 + {}^{n-1} C_1 + \dots + {}^{n-1} C_{n-1}]$

$= n \cdot 2^{n-1}$

MATHS FOR JEE MAINS & ADVANCED

Aliter : (Using method of differentiation)

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n \quad \dots\dots\dots(A)$$

Differentiating (A), we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + n.C_nx^{n-1}.$$

Put $x = 1$,

$$C_1 + 2C_2 + 3C_3 + \dots + n.C_n = n.2^{n-1}$$

(ii)
$$\begin{aligned} \text{L.H.S.} &= \sum_{r=0}^n \frac{C_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^n \frac{n+1}{r+1} {}^nC_r \\ &= \frac{1}{n+1} \sum_{r=0}^n {}^{n+1}C_{r+1} = \frac{1}{n+1} [{}^{n+1}C_1 + {}^{n+1}C_2 + \dots + {}^{n+1}C_{n+1}] = \frac{1}{n+1} [2^{n+1} - 1] \end{aligned}$$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \quad (\text{where } C \text{ is a constant})$$

Put $x = 0$, we get, $C = -\frac{1}{n+1}$

$$\therefore \frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}$$

Put $x = 1$, we get

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

Put $x = -1$, we get

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots = \frac{1}{n+1}$$

Ex. Prove that $C_1 - C_3 + C_5 - \dots = 2^{n/2} \sin \frac{n\pi}{4}$.

Sol. Consider the expansion $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \quad \dots(i)$

putting $x = -i$ in (i) we get

$$(1-i)^n = C_0 - C_1i - C_2 + C_3i + C_4 + \dots + (-1)^n C_n i^n$$

or
$$2^{n/2} \left[\cos\left(-\frac{n\pi}{4}\right) + i \sin\left(-\frac{n\pi}{4}\right) \right] = (C_0 - C_2 + C_4 - \dots) - i(C_1 - C_3 + C_5 - \dots) \quad \dots(ii)$$

Equating the imaginary part in (ii) we get $C_1 - C_3 + C_5 - \dots = 2^{n/2} \sin \frac{n\pi}{4}$.

Ex. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ then prove that $\sum_{0 \leq i < j \leq n} (C_i + C_j)^2 = (n - 1)^{2n}C_n + 2^{2n}$

Sol. L.H.S $\sum_{0 \leq i < j \leq n} (C_i + C_j)^2$
 $= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \dots + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \dots$
 $\quad + (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \dots + (C_2 + C_n)^2 + \dots + (C_{n-1} + C_n)^2$
 $= n(C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2) + 2 \sum_{0 \leq i < j \leq n} C_i C_j$
 $= n \cdot {}^{2n}C_n + 2 \cdot \left\{ 2^{2n-1} - \frac{2n!}{2 \cdot n!n!} \right\} \Rightarrow n \cdot {}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n - 1) \cdot {}^{2n}C_n + 2^{2n} = \text{R.H.S.}$

Multinomial Theorem

Using Binomial theorem, we have $(x + a)^n = \sum_{r=0}^n {}^n C_r x^{n-r} a^r, n \in N$
 $= \sum_{r=0}^n \frac{n!}{(n-r)!r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r, \text{ where } s + r = n$

This result can be generalized in the following form.

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{r_1+r_2+\dots+r_k=n} \frac{n!}{r_1!r_2!\dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!\dots r_k!} x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + \dots + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular Cases

(i) $(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Ex. Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Sol. $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!r_2!r_3!r_4!} (a)^{r_1} (-b)^{r_2} (-c)^{r_3} (d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

\therefore coeff. of $a^2 b^3 c^4 d$ is $\frac{(10)!}{2!3!4!1!} (-1)^3 (-1)^4 = -12600$

MATHS FOR JEE MAINS & ADVANCED

Ex. Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3 .

Sol. Total number of terms = $^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

$$\text{Coefficient of } x^2y^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$$

BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

If $n \in \mathbb{Q}$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \infty$ provided $|x| < 1$.

- (i) When the index n is a positive integer the number of terms in the expansion of $(1 + x)^n$ is finite i.e. $(n+1)$ & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$
- (ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1 + x)^n$ is infinite and the symbol nC_r cannot be used to denote the coefficient of the general term.
- (iii) Following expansion should be remembered ($|x| < 1$).
- (a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots \infty$ (b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots \infty$
- (c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \infty$ (d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \infty$
- (e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + \frac{(-1)^r (r+1)(r+2)}{2!}x^r + \dots$
- (f) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{2!}x^r + \dots$
- (iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

Approximations

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately.
This is an approximate value of $(1 + x)^n$

Ex. Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Sol. $(r + 1)^{\text{th}}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$\begin{aligned} T_{r+1} &= \frac{-n(-n-1)(-n-2)\dots(-n-r+1)}{r!} (-x)^r \\ &= (-1)^r \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} (-x)^r = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r \\ &= \frac{(n-1)!n(n+1)\dots(n+r-1)}{(n-1)!r!} x^r \end{aligned}$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!r!} = {}^{n+r-1}C_r$ Proved

Ex. If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}}$

Sol.
$$\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1 - \frac{3}{2}x + 1 - \frac{5x}{3}}{2\left(1 + \frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 + \frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2 - \frac{x}{4} - \frac{19}{6}x\right) = 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$$

EXPONENTIAL SERIES

- (a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.
- (b) Logarithms to the base ‘ e ’ are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm**.

(c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$; where x may be any real or complex number & $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$

(d) $a^x = 1 + \frac{x}{1!} \ln a + \frac{x^2}{2!} \ln^2 a + \frac{x^3}{3!} \ln^3 a + \dots \infty$, where $a > 0$

(e) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \infty$

Ex. Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x .

Sol. In the exponential series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Hence $e^{2x+3} = e^3 \cdot e^{2x} = e^3 \left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots \right]$

Thus, the coefficient of x^2 in the expansion of e^{2x+3} is $e^3 \cdot \frac{2^2}{2!} = 2e^3$.

LOGARITHMIC SERIES

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$, where $-1 < x \leq 1$ (b) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$, where $-1 \leq x < 1$

❖ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty = \ln 2$ ❖ $e^{\ln x} = x$; for all $x > 0$ ❖ $\ln 2 = 0.693$ ❖ $\ln 10 = 2.303$

Ex. If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e(1 + px + qx^2) = (\alpha + \beta)x - \frac{\alpha^2 + \beta^2}{2}x^2 + \frac{\alpha^3 + \beta^3}{3}x^3 - \dots$$

Sol. Right hand side = $\left[\alpha x - \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} - \dots \right] + \left[\beta x - \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} - \dots \right]$

= $\log_e(1 + \alpha x) + \log_e(1 + \beta x)$ = $\log_e(1 + (\alpha + \beta)x + \alpha\beta x^2)$ = $\log_e(1 + px + qx^2)$ = Left hand side.

Here, we have used the facts $\alpha + \beta = p$ and $\alpha\beta = q$. We know this from the given roots of the quadratic equation. We have also assumed that both $|\alpha \cdot x| < 1$ and $|\beta x| < 1$.

TIPS & FORMULAS

If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then : $(x + y)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1} y + {}^n C_2 x^{n-2} y^2 + \dots + {}^n C_r x^{n-r} y^r + \dots + {}^n C_n y^n = \sum_{r=0}^n {}^n C_r x^{n-r} y^r$

1. Important Terms in the Binomial Expansion

(a) **General term** : The general terms or the $(r + 1)$ th term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^n C_r x^{n-r} \cdot y^r$$

(b) Middle Term

The middle term (s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle terms which is given by $T_{(n+2)/2} = {}^n C_{n/2} \cdot x^{n/2} \cdot y^{n/2}$.

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(c) Term Independent of x

Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

2. If $(\sqrt{A} + B)^n = I + f$, where I & n are positive integers & $0 \leq f < 1$, then

(a) $(I + f) \cdot f = K^n$ if n is odd & $A - B^2 = K > 0$

(b) $(I + f)(1 - f) = k^n$ if n is even & $\sqrt{A} - B < 0$

3. Some Results on Binomial Coefficients

(a) ${}^n C_x = {}^n C_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^n C_{r-1} + {}^n C_r = {}^{n+1} C_r$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \dots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = 2^n C_n = \frac{(2n)!}{n!n!}$

(h) $C_0 \cdot C_r + C_1 \cdot C_{r+1} + C_2 \cdot C_{r+2} + \dots + C_{n-r} \cdot C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

Remember : $(2n)! = 2^n \cdot n! [1.3.5 \dots (2n-1)]$

4. Greatest Coefficient & Greatest Term in Expansion of $(x + a)^n$

(a) If n is even greatest coefficient is ${}^n C_{n/2}$

If n is odd greatest coefficient is $\frac{{}^n C_{n-1}}{2}$ or $\frac{{}^n C_{n+1}}{2}$

(b) For greatest term :

$$\text{Greatest term} = \begin{cases} T_p \ \& \ T_{p+1} & \text{if } \frac{n+1}{\left| \frac{x}{a} \right| + 1} \text{ is an integer} \\ T_{q+1} & \text{if } \frac{n+1}{\left| \frac{x}{a} \right| + 1} \text{ is non integer and } \in (q, q+1), q \in I \end{cases}$$

5. Binomial Theorem for Negative or Fractional Indices

If $n \in \mathbb{Q}$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \infty$ provided $|x| < 1$

Note

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots \infty$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \infty$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \infty$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \infty$

6. Exponential Series

(a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$; where x may be any real or complex number & $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$

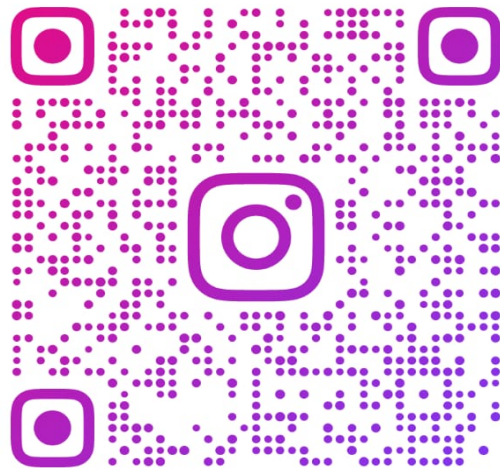
(b) $a^x = 1 + \frac{x}{1!} \ln a + \frac{x^2}{2!} \ln^2 a + \frac{x^3}{3!} \ln^3 a + \dots \infty$, where $a > 0$

7. Logarithmic Series

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$, where $-1 < x \leq 1$

(b) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \infty$, where $-1 \leq x < 1$

(c) $\ln \frac{(1+x)}{(1-x)} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty \right) | x | < 1$



@IIT_GURUKUL_OFFICIAL